Real or Synthetic Securities: Does it Make any Difference

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ABSTRACT: *We present a model of a stock market with three periods of time, one stock and one risk-free asset, and three kinds of agents: Buy and hold, portfolio insurers and market makers. Portfolio insurers buy synthetic put options on the stock. We show that in this model, the information about the amount of synthetic puts being transacted changes the market equilibrium and the price of the put. We compare the difference in this information when there exists a real put options market and when there*

1. Introduction

The advances in finantial theory in the last two decades[1] have created the foundations for evaluating real securities. Real securities can be replicated by trading strategies in either securities and risk-free asset. But this method of evaluating real securities has made possible to trade derived securities even if there are no real securities. These securities that are traded but have no real counterpart are called synthetic securities. These theoretical advances have had enourmous impact in the securities markets, as the growth of importance of option markets, interest rate, futures, etc., are a vivid illustration.

It has been a well accepted belief that there are no real differences for the market if a traded security is real or synthetic. Recently Grossman [1988] have challenged this belief. He argues that real securities have an important signalling function that can not be fulfilled by synthetic securities. If future price volatility is correlated with the volume of dynamic hedging strategies (which

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[1] Black and Scholes [1973] is the seminal work.

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typically involves trading in derived securities) then there will be less information in the economy if the traders use synthetic rather than real securities in their strategies. To help understanding Grossman's argument let's focus on a very common form of portfolio insurance: stock put options. If the put option is heavily traded as synthetic put this will typically imply great demand (positive or negative) for the stock in the comming periods due to the management of portfolio associated with the synthetic puts, compared with the demand by low use of synthetic put options. This variability in the demand due to the use of synthetic put options will enhance the basic volatility of the price of the stock associated with the put. However this enhanced volatility could be dampened if it was known the demand for put options (which is the case if there is a real put option) basically for two reasons:

(a) market makers, knowing that stock price in the coming periods will have enhanced volatility (which creates larger deviations of stock price to its fundamental value) might bring more money to the market in the coming periods which somewhat offsets the demand generated by portfolio insurers and reduces volatility; and

(b) portfolio insurers, knowing the real costs of the insurance might reduce their demand for insurance, which also dampens the variability of portfolio insurers demand, and therefore stock price volatility in the coming periods.

Grossman's paper have received skeptical appraisal by economists. As we have already the opportunity to comment, it was a common belief of economists that there were no difference for the market if a traded security is real or synthetic. The argument can be roughly described as the following: since derived securities are a function of existing securities and a risk free asset, it could not reveal information that there were not already revealed by the underlying securities and risk-free asset. Although this argument seems quite compelling, it is nevertheless wrong. The reason is the following. Securities prices today are a function of random variables, which will unfold in the future periods. The put price is also a function of these random variables, but a *different* function.

We can see why intuitively. The put formula for the binomial case is $P = SA + B$, where P is the put price, S is the current stock- price, and A and B are respectively the amount of stock and the quantity of dollars in risk-free asset needed to replicate the put. If A is constant the real put could not reveal any additional information that was not already transmitted by the stock price. However A depends on the future stock price distribution in a way which may not be fully captured by the current stock price. Therefore P may indeed reveal additional information.

If a synthetic put is heavily transacted., the hedging strategies planned today will distort the prices tomorrow in a way that was not planned. Since the synthetic putt's price today depends on the expected prices for tomorrow and they can not be properly evaluated by the insurers[2], the synthetic putt's price calculated today will not correspond to the price of the hedging strategy (the price of a real put). This will not be the case if a real put is transacted. Because in that case the real put price as seen in the market today gives information about the securities expected prices for tomorrow, which already reflects the effect of the hedging strategies.

Finally, as we have seen, the amount θ of put options being synthesised influences the price of real and synthetic securities, i.e. the events of different amounts θ are revelant for the market. The use of only synthetic options contributes to make the market incomplete in such way that a real option is no longer redundant.[3]

2. The Model

The economy has three periods denoted by the numbers 1, 2 and 3, respectively. There is one unity of the stock to be transacted (although short selling is also allowed). The price of the stock at period 1 is denoted by S_1.

Demand (and supply) for the stock have three components. The first component is the net demand of buy and hold investors, who initially hold the market. Their demand is described as X_1. The second component is the net demand of portfolio insurers. Portfolio insurers want to buy some quantity of stocks and the same quantity of a (synthetic or real) put with striking price K. K is public known.

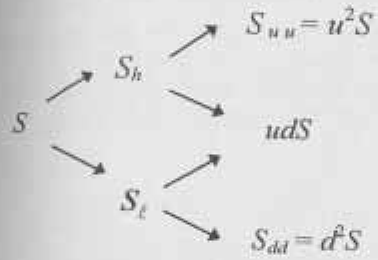
[2] Because since they don't know the amount of synthetic puts being transacted, they can not evaluate the distortion on the security prices tomorrow due to their hedging strategies.

[3] This incompleteness arises not from an excess of "external states of nature" but from an excess of states of nature coming internally from the market. Namely, the amount θ becomes a (continuous) state of nature (in the sense that it influences the prices of stocks), which could be non-observable.

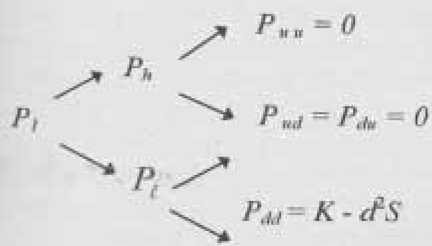
Their demand is described as Y_1. Finally, the third component is the net demand of the market makers. They are assumed to have, the utility function $\mathcal{U}(x) = -e^{-ax}$, and to maximise their expected utility at period 3. Their net demand is described as Z_1. Before the period one market opens, buy and hold investors, who initially hold the market (composed of one stock) decide α, the fraction of the market they want to hold until period three unfolds, which is a random variable with known probability. Therefore $X_1 = \alpha - 1$. Also before the period 1 market opens. portfolio insurers decide θ, the fraction of the market that they want to insure by buying put options, which is also a random variable with known probability. Therefore $Y_1 = \theta(1 + A_1)$, where A_1 is the amount of stocks required to synthesise the option. We call r the rate of return (one plus the rate of interest) that the risk-free asset pays on each period and which is also the opportunity cost of the money the market makers spend in investing on stocks for one period, $r > 1$.

At period 2 some information arrives about what will be the fundamental value of the stock in the last period. This information can take two values, with known probabilities. The first, denoted by h ("high") implies that the stock price in period 3 will assume one of two values, with known probabilities: u^2S or udS, where u, d and S are strictly positive real numbers such that $u > r > 1 > d$. And the second, ℓ ("low") implies that the stock price in period 3 will assume one of two values, with known probabilities: udS or d^2S. The information is h with probability p and it is ℓ with probability $(1-p)$. After the arrival of information about period 3's likelihood of the different states, the period two market opens and transactionsare made. Buy and hold investors don't come to the second period market. The demand of portfolio insurers is $Y_j = \theta\left(A_j - A_1\right), j = h, \ell$; where A_j is the amount of stock required at period 2, $j = h, \ell$, in the dynamic strategy which synthesises the put. The market makers' second period demand is represented by Z_j, $j = h, \ell$.

At period 3 all the uncertainty is resolved and payoffs are made to buy and hold investors. portfolio insurance investors and market makers. Independently of the information at period two, the probability that the price of stock rises from period two to period 3 is p. We can represent the evolution of the market for stocks by the following diagram



Corresponding to this diagram, there is a diagram which represents the (synthetic or real) put price at the striking price $udS < K < d^2S$.[4]



For the sake of clarification we will represent all the above decision and pre-determined variables in the following table

Period 1		Period 2		Period 3
Before Mkts. Open	After Mtks. Open	Before Mtks. Open	After Mtks. open	After Mtks. Open
α (random) θ (random)	X_1, Y_1, Z_1 S_1, P_1	ℓ or h	Y_j, Z_j S_j, P_j	either u^2S udS ddS

We analyse different cases:

[4] We assume that K falls in this region in order to *avoid algebraic inconveniences.*

Case I

Agents believe that $\theta \approx \theta_0$ with probability 1. For example $\theta = 0$ or $0 \leq \theta << 1$ if the agents expect that there are not synthetic puts being transacted or that the amount of synthetic puts is negligible with respect to the whole market. Agents expected that with probability 1, $S_h = uS$, $S_l = dS$. The values of u and d are assumed to satisfy

$$p(u - r) \geq (1 - p)(r - d).$$

So that the expected return for holding a stock from period two to period three, $p(u - r) - (1 - p)(r - d)$, is non-negative. This will imply that $Z_1 \geq 0$. They are also assumed to satisfy the following expected equilibrium conditions:

$$\alpha - 1 + \theta_0 (1 + A_1 (uS, dS)) + Z_1(S) = O$$

$$\theta_0 (A_h - A_1 (u S, dS)) = Z_h (u S)$$
$$\theta_0 (A_l - A_1 (u S, dS)) = Z_l (dS).$$

So that the expected prices uS, dS, for period 2 are consistent with the expected value of θ. The quantities a and θ are supposed to be random variables related by the formula

(1.1) $$\alpha - 1 + \theta (1 + A_1) = \Delta,$$

where Δ is a constant. The stock price S at period 1 is determined by the equilibrium equation

$$\alpha - 1 + \theta (1 + A_1) + Z_1(S) = 0$$

If the striking price of the put is $\bar{K} = kS$ with fixed k exogenously we have that

$$A_1 = A_1(uS, dS) = - \frac{P_\ell}{uS - dS} = - \frac{1}{r} t$$

(see §5 and §2). In particular A_1 does not depend on S and hence the equilibrium equation is equivalent to Z_1 (S) = $-\Delta$. This gives a constant price S for the various values of θ, so that the agents can not infer the value of θ by just observing S.[5]

In this case we can prove that $S_h(\theta)$ increases with θ, $S_\ell(\theta)$ decreases with θ and that the real value of the put option, $P_1(\theta)$, increases with θ despite of its price at period 1 being fixed at PI (θ_o).

Case II

In this case portfolio insurers know the value of θ but the market makers do not know the amount of portfolio insurance. Market markers believe that $\theta = \theta_o$ and $S_h = uS$, $S_\ell = dS$ (which is the case when $\theta = \theta_o$), with probability 1. Portfolio insurers can synthesise correctly the put and calculate the exact cost of their insurance. Nevertheless it is known by portfolio insurers we can prove that the volatility of the price of the stock increases with the amount θ of insurance. We also assume in this case that α and f are random variables related in such way that the price S_1 remains constant, so that market makers can not infer θ by observing S_1. The assumption is:

$$1 - \alpha + \theta(1 + A_1(S_h(\theta), S_\ell(\theta))) = \Delta,$$

for a constant Δ such that $Z_1(S, u, d) = Z_1(S) = \Delta$. Market makers expect $\theta = \theta_0$ and $S_h = uS$, $S_\ell = dS$ (which is the case when $\theta = \theta_o$).

Case III

In this case neither the portfolio insurers nor the market makers know the value of θ. But they correctly believe that θ is a random variable with probability distribution μ on $[O, +\infty[$. The market becomes incomplete and portfolio insurers can not

[5] We want to stress that the assumption (1.1) is made only for S not to reveal perfectly θ to the market participants.

synthesise correctly the put. We see that in this case $S_h(\theta)$ increases with θ and $S_\ell(\theta)$ decreases with θ.

Case IV

In this case the portfolio insurers are assumed to know the value of θ, but the market makers do not. The market makers correctly believe that θ is a random variable with probability distribution μ. We see that $S_h(\theta)$ increases with θ and $S_\ell(\theta)$ decreases with θ.

Case V

In this case θ is supposed to be public known. This case corresponds to the existence of a real put if we assume that all the suppliers of the put synthesise it in order to cover their positions. Then the value of θ is seen as the amount of written puts in the options market. Due to technical difficulties we don't prove the same as in the other cases. We observe that compared to case IV, the prices S_h and S_ℓ are either both greater than those of case IV or are both lower than those of case IV, depending on the strategy at period 1 of the market makers.

Existence of a Real Put.

At the end of the paper we show how the mathematical analysis on the cases above can be applied to situations in which there exists a real put option.

3. Price of the option

Consider a European option[6] with striking price K and which matures at period 3. Then its value at period 3 is known to be $P_3 = (K - S_3)^+ := max\{0, K - S_3\}$ if it is a put option and $P_3 = (S_3 - K)^+$ if it is a call option[7].

Let P_j, ($j = h, l$) represent the option price in period 2 when the information is "high" and "low" respectively. Let A_j, B_j be the amount of stock and risk-free asset required to synthesise one unit of the option in period 2 when the information is j. Let P_1, A_1, B_1, denote the option price, the amount of stock and the amount of risk-free asset required to synthesise one unit of the option in period 1. We know that $P_h = S_h A_h + B_h$, and that $P_{uu} = u^2 SAh + rB_h$, and $P_{ud} = udSA_h + rB_h$. Performing the necessary calculations we arrive at

$$A_h = \frac{Puu - Pud}{(u-d)uS} \qquad B_h = \frac{uP_{ud} - dP_{uu}}{(u-d)r}$$

$$P_h = 1\ (\lambda_h P_{ud} + (1 - \lambda_h)\ P_{uu}) \text{ where } \lambda_h = \left(u - \frac{rS_h}{uS}\right)\frac{1}{(u-d\ )}$$

similarly

(2.1) $$A_\ell = \frac{P_{ud} - P_{dd}}{(u-d\ )dS};\quad B_\ell = \frac{uP_{dd} - dP_{ud}}{(u-d\ )r}$$

(2.2) $$P_\ell = \frac{1}{r}\left(\lambda_\ell P_{dd} + (1-\lambda_\ell)P_{ud}\right)\ ;\lambda_\ell = \left(u - \frac{rS_\ell}{dS}\right)\frac{1}{(u-d)}$$

(2.3) $$P_1 = \frac{1}{r^2}[\eta\lambda_\ell P_{dd} + (1-\lambda_\ell) + (1-\eta)\lambda_\ell)P_{ud} + (1-\eta)(1-\lambda_\ell)P_{uu}]$$

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6 See Black and Scholes [1973].

7 S3 is the stock price at period 3.

(2.4) $$\eta = \frac{S_h - rS}{S_h - S_\ell}$$

If the expected conditions were realised (i.e. $S_h = uS$, $S_\ell = dS$), we would have

$$P_h^* = \frac{1}{r}[qP_{du} + (1-q)P_{uu}] \quad ; \quad P_\ell^* = \frac{1}{r}[qP_{dd} + (1-q)P_{ud}]$$

$$P_1 = \frac{1}{r^2}[q^2P_{dd} + 2q(1-q)P_{ud} + (1-q)^2 P_{uu}]$$

$$A_1 = \frac{-P_\ell}{S_h - S_\ell}$$

Where $q = \frac{u-r}{u-d}$

4. *The Behaviour of the Agents*

Buy and hold investors net demand is $X_1 = \alpha - 1$ in the first period. They do not come to the market in the second period. The demand of portfolio insurance investors is $Y_1 = \theta(1 + A_1)$ of stocks in the first period. In the second period they demand $Y_j = \theta(A_j - A_1)$ if the information is $j = h, \ell$. Note that A_1 can only depend on the *expected* prices for period 2, S_h^* S_ℓ^*, as seen at period 1; and the A_j's depend on the expected prices for period 3 which we assume that are given by fundamentals, in particular they do not depend on the realised prices S_j at period 2 and hence they are kept fixed in this model.

We assume that market makers have the risk-adverse utility function $U(w + x) = -e^{-a(w+x)}$ where $w + x$ is the total wealth of the market makers and x is the amount of money being transacted in the relevant periods. Assume that they have no shares of stock before the market opens at period 1. Let D_j, $j = h, \ell$ be the amount of shares of the stock that they are willing to own at period 2, so that $Z_j = D_j - Z_1$ for $j = h, \ell$.

We first calculate their demand at period 2. Their choice comes out of the maximisation of the expected profits: $\theta_j(Z_1, D_j) = \max_x \theta_j(Z_1, x)$.

$$f_j(Z_1, x) := \sum_{k=u,d} p_k\, u\left(w + Z_1\left(S_j - rS\right) + x\left(S_{jk} - rS_j\right)\right)$$
$$= e^{-aw} e^{-aZ_1 r(S_j - rS)} \sum_k p_k\, u\left(x\left(S_{jk} - rS_j\right)\right)$$

(3.1)

$$D_j = \frac{1}{a(u-d)S_j^*} \log\left(\frac{p\left(S_{ju} - rS_j\right)}{(1-p)\left(rS_j - S_{jd}\right)}\right), \quad S_j^* = \begin{cases} uS & j = h \\ dS & j = \ell \end{cases}$$

In particular, D_j does not depend on Z_1. For period 1 they solve the problem: $f(Z_1, D_h, D_\ell)$ max $f(x, D_h, D_\ell)$, where

$$f(x, D_h, D_\ell) = \sum_{j,k} p_{jk}\, u\left(\omega + xr\left(S_j - rS\right) + D_j\left(S_{jk} - rS_j\right)\right)$$
$$= e^{-a\omega} \sum_j p_j\, e^{-ar(S_j - rS)x} \left(\sum_k p_k\, u\left(D_j\left(S_{jk} - rS_j\right)\right)\right)$$

On case I we are assuming that they believe that $\theta = \theta_o$, they use the expected values $S_h^* = uS$, $S_\ell^* = dS$, $D_j^* = D_j\left(S_j^*\right)$, $j = h, \lambda$. Then

$$R_k := D_j^*\left(S_{jk} - rS_j^*\right) = \frac{k-r}{a(u-d)}\log\left(\frac{p(u-r)}{(1-p)(r-d)}\right), \quad k = u,d$$

does not depend on j and

$$f(x, D_j^*, D_\ell^*) = e^{-a\omega}\left(\sum P_l e^{-ar(Sj-rS)x}\right)\left(\sum_k p_k u(R_k)\right).$$

Hence the demand Z_1, is obtained by maximising the firs factor:

$$Z_1^* = \frac{1}{ar(u-d)S}\log\left(\frac{p(u-r)}{(1-p)(r-d)}\right).$$

Note that Z_1 in this case is the same as if there were no thirc period, reduced by the interest rate payed for holding the profits from period 2 to period 3. Now we can calculate the demand at period 2:

$$Z_j = D_j - Z_1^*$$

$$= \frac{1}{a(1-d)}\left[\frac{1}{S_j^*}\log\left(\frac{\left(p(uS*_j - rS_j\right)}{(1-p)\left(rS_j - dS_j^*\right)}\right) - \frac{1}{rS}\log Q\right]$$

$$Q = \frac{p(u-r)}{(1-p)(r-d)}.$$

5. Case I: Equilibrium when the amount of insurance is Unknown.

Equilibrium at period 1 requires

$$\alpha - 1 + \theta(1 + A_1) + Z_1 = O$$

which gives the initial price S. We consider α and θ as being random variables related by the formula[8]

$$\alpha - 1 + \theta(1 + A_1) + Z_1(S) = O.$$

Equilibrium at period 2 is given by

$$Z_j\left(S_j\right) = -\theta\left(A_j - A_1\right) \qquad j = k,\ \ell.$$

Where A_j, $j = k,\ \ell$, do not depend on S_j (see § 2). Solving this equation for S_j gives

$$S_j = \frac{1}{r}\left(\frac{pu + (1-p)\Delta_j(\theta)d}{p + (1-p)\Delta_j(\theta)}\right)S_j^*$$

(4.1) $$\Delta_j(\theta) = Q^{\frac{j}{r}} \exp\left(-\theta\alpha(u-d)S_j^* A_j\right) > 0 \quad j = u,d$$

$$A_j := A_j - A_1 \quad , \qquad Q = \frac{p(u-r)}{(1-p)(r-d)}.$$

We have

(4.2) $$\frac{\partial S_j}{\partial \theta} = -\left(\frac{\partial \Delta_j}{\partial \theta}\right)\left(\frac{(1-p)p(u-d)}{r}\right)\left(\frac{1}{p+(1-p)\Delta_j(\theta)}\right)^2 S_j^*$$

8 The role played by the random variable α is to blur the information about θ that market makers could infer from observation of S1.

(4.3) $$\frac{\partial \Delta j(\theta)}{\partial \theta} = -Aj\left(a(u-d)S_j^*\right)\Delta_j(\theta)$$

$\dfrac{\partial \Delta_j(\theta)}{\partial \theta}$ is proportional to $-A_j D_j(\theta)$.

6. The Case of a Synthetic Put option

Suppose that portfolio insurers an European put option with striking price $K = \bar{k}S$ $\left(d^2S < K < udS\right)$ which matures at period 3. Then $P_{dd} = K - d^2S$ and $P_{ud} = P_{du} = P_{uu} = 0$. Therefore $P_h = 0$ and

$$P_\ell = \frac{1}{r}(\theta)\left(K - d^2S\right) = \frac{1}{r}\lambda(\theta)\left(\bar{k} - d^2\right)S$$

(5.1) $$P_1 = \frac{1}{r^2}\eta(\theta)\lambda\left(K - d^2S\right) = \frac{1}{r}\eta(\theta)\lambda(\theta)\left(\bar{k} - d^2\right)S$$

where $\eta(\theta) = \eta$ and $\lambda(\theta) = \lambda_\ell$ are given by (2.2) and (2.4) for $S_j = S_j(\theta)$, $j = h, \ell$ as in (4.1). Also

$$A_h = 0, \qquad A_\ell = \frac{-P_{dd}}{(u-d)dS}$$

since portfolio insurers do not know θ, they calculate $A1$ with the expected prices S^*_j.

$$A_1 = \frac{-P_\ell\left(S_h^*, S_\ell^*\right)}{(u-d)S} = \frac{(u-r)P_{dd}}{(u-d)^2 S}$$

$$A_h = -A_1 \qquad A_\ell = A_\ell - A_1 = -\left(\frac{1}{d} - \frac{u-r}{u-d}\right)\frac{P_{dd}}{(u-d)S} < 0.$$

Using (4.2) we see that

$$\frac{\partial S_h}{\partial \theta} > 0, \frac{\partial S_\ell}{\partial \theta} < 0 \quad \text{for all} \quad \theta \geq 0$$

Also, the actual price of the insurance $P_1(\theta)$ varies with θ :

for

$$\overset{\circ}{\eta} := \frac{\partial \eta}{\partial \theta}, \quad \overset{\circ}{\lambda} := \frac{\partial \lambda}{\partial \theta}, \quad \overset{\circ}{S}_\ell = \frac{\partial S_\ell}{\partial \theta}, \quad \overset{\circ}{S}_h = \frac{\partial S_h}{\partial \theta}, \text{ we have:}$$

$$\frac{\partial P_1}{\partial \theta} = \frac{1}{r^2}\left(\overset{\circ}{\eta}\lambda + \eta\overset{\circ}{\lambda}\right)P_{dd}$$

$$\overset{\circ}{\lambda} = \frac{-r\overset{\circ}{S}_\ell}{(u-d)ds} > 0, \qquad \lambda = \frac{udS - rS_\ell}{(u-d)dS}$$

$$\overset{\circ}{\eta} = \frac{\overset{\circ}{S}_h}{S_h - S_\ell} - \frac{S_h - rS}{S_h - S_\ell}\;\frac{\overset{\circ}{S}_h - \overset{\circ}{S}_\ell}{S_h - S_\ell}$$

$$(5.2) \qquad = \frac{\overset{\circ}{S}_h}{S_h - S_\ell}(1-\eta) + \frac{\overset{\circ}{S}_\ell}{S_h - S_\ell}\eta$$

but for $\theta \geq \theta_\circ$, $0 < \dfrac{S_h - rS}{S_h - S_\ell} = n < 1$, because

$S_\ell(\theta) \leq S_\ell(\theta) = dS < rS < uS = S_h(\theta_\circ)$ for $\theta \geq \theta_\circ$. Hence for $\theta \geq \theta_\circ$ the first term in (5.2) is positive and

$$\overset{\circ}{\eta}\lambda + \eta\overset{\circ}{\lambda} > \eta\overset{\circ}{S}_\ell\left(\frac{\lambda}{S_h - S_\ell} - \frac{r}{(u-d)dS}\right)$$

$$\frac{\lambda}{S_h - S_\ell} = \frac{uds - rS_\ell}{(u-d)dS}\left(\frac{1}{S_h - S_\ell}\right) = \frac{r}{(u-d)dS}\left(\frac{udS - rS_\ell}{r(S_h - S_\ell)}\right),$$

but for $\theta \geq \theta_0$, $rS_h \geq ruS \geq rudS \geq udS$, so that the last factor is < 1 and then $\frac{\lambda}{S_h - S_\ell} < \frac{r}{(u-d)dS}$. Since $\overset{\circ}{S}_\ell < 0$, we have that $\overset{\circ}{\eta}\lambda + \eta\overset{\circ}{\lambda} > 0$ and then

$$\frac{\partial P_1}{\partial \theta} > 0, \quad \text{for} \quad \theta \geq \theta_0,$$

i.e. the real price of the insurance increases with the amount of portfolio insurance when the level of portfolio insurance is above the expected value. If portfolio insurers would know θ, they could have changed their strategies. For example, suppose that each portfolio insurer decides to invest his capital with a full insurance buying puts with striking price $K = S$. If σS is the total amount of capital that is going to be invested, we have that $\theta(S + P_1(\theta)) = \sigma S$ and that

$$\theta = \frac{\sigma S}{S + P_1(\theta)} < \frac{\sigma S}{K + P_1(\theta_0)}, \qquad \text{for } \theta \geq \theta_0,$$

i.e. they would buy less puts than if they thought that $\theta = \theta_0$.

7. Case II: If the Amount of Insurance is Known only by the Portfolio Insurers and the Market Makers Believe that $\theta = \theta_0$ with Probability 1.

Now suppose that θ is known by the portfolio insurers and the market makers believe that $\theta = \theta_0$ with probability 1. The demand function of the market makers at period 2 is the same as in case I. Also $A_\ell, B_\ell, A_h, = 0 = B_h$ remain the same as in case I. but $A_1 = A_1(S_h(\theta), S_\ell(\theta))$ changes. We assume that the demand α - 1 at period 1 of buy and hold agents is a random variable related to θ by the formula

(6.0) $$\alpha - 1 + \theta\left(1 + A_1\left(S_h(\theta), S_\ell(\theta)\right)\right) = \Delta\,(\theta)) = \Delta$$

where Δ is a constant. So that the price S of the stocks at period 1, which is given by $Z_1\left(S; S_h^* = uS, S_\ell^* = dS\right) = -\Delta$, does not reveal θ.

$$A_1 = A_1(\theta) = \frac{-P_\ell}{S_h - S_\ell} = \frac{-A_\ell S_\ell - B_\ell}{S_h - S_\ell}$$

$$A_h = -A_1, \qquad\qquad A_\ell = A_\ell - A_1$$

From equations (4.2) and (4.1), we get

$$\frac{\partial Sj}{\partial} = -\frac{\partial \Delta}{\partial \theta} K_j^1, \qquad \text{where } K_j^1 = K_j^1(\theta) > 0, \; j = k, \ell.$$

$$\frac{\partial \Delta j}{\partial} = -\frac{\partial}{\partial \theta}(\theta A_j) K_j^2, \quad \text{where } K_j^2 = K_j^2(f) > 0, \; j = k, \ell.$$

$$\frac{\partial S_j}{\partial} = \left(\mathrm{A}_j - \theta \frac{\partial A_1}{\partial \theta}\right) K_j, \qquad K := K_j^1 K_j^2 > 0.$$

We shall prove that $\overset{\circ}{S}_h(\theta) := \left.\frac{\partial S_h}{\partial \theta}\right|_\theta > 0$ and $\overset{\circ}{S}_h(\theta) := \left.\frac{\partial S_\ell}{\partial \theta}\right|_\theta < 0$ for all $\theta \geq \theta_o$, so that $S_h(\theta)$ increases and $S_\ell(\theta)$ decreases with θ and in particular the volatility $S_h(\theta)$ - $S_\ell(\theta)$ increases with the amount of insurance. The same argument as in §5 show that $P_1(\theta)$ increases with θ, for $\theta \geq \theta_o$, in this case.

From (4.4) we have that $P_\ell = A_\ell S_\ell(\theta) + B_\ell > 0$ for all θ, and also that

$$A_1(\theta) = \frac{-P_\ell}{S_h(\theta) - S_\ell(\theta)} < 0.$$

With this we see that $\left.\frac{\partial S_h}{\partial\theta}\right|_{\theta=0} = -A_1(\theta=0)\cdot K_h > 0$. We want to prove that $\dot{S}_h(\theta) > 0$ for all $\theta \geq 0$. Suppose that there exists some $\theta > 0$ such that $\dot{S}_h(\theta) = 0$, let $\theta_1 > 0$ be the least such θ. Then $\left.\frac{\partial}{\partial\theta}(\theta A_h)\right|_{\theta_1} = 0$ and hence $\left.\frac{\partial}{\partial\theta}(\theta A_h)\right|_{\theta_1} = A_\ell + \frac{\partial}{\partial\theta}(\theta A_h) = A_\ell < 0$. So that

$$(6.1) \qquad \dot{S}_\ell(\theta_1) = K\ell \left.\frac{\partial}{\partial\theta}(\theta A_h)\right|_{\theta_1} < 0$$

We have

$$-\left.\frac{\partial}{\partial\theta}(\theta A_h)\right|_{\theta_1} = A_1(\theta_1) + \left.\frac{\partial A_1}{\partial\theta}\right|_{\theta_1}$$

$$A_1(\theta_1) = \theta_1 \left.\frac{\partial}{\partial\theta}\left(\frac{A_\ell S_\ell(\theta) + B_\ell}{S_h(\theta) - S_\ell(\theta)}\right)\right|_{\theta_1}$$

$$A_1(S_h - S_\ell)^2 = \theta A_\ell \dot{S}_\ell(S_h - S_\ell) - \theta(A_\ell S_\ell + B_\ell)\left(\dot{S}_h - \dot{S}_\ell\right) = \theta \dot{S}_\ell(A_\ell S_h) + B_\ell$$

because $\dot{S}(\theta) = 0$. Therefore

$$(6.2) \qquad \dot{S}_\ell(\theta_1) = \frac{A_1(S_h - S_\ell)^2}{\theta(A_\ell S_h + B_\ell)}$$

but

$$A_h S_h(\theta_1) + B_\ell = -\frac{P_{dd}}{(u-d)dS} S_h(\theta_1) + \frac{uP_{dd}}{(u-d)r}$$

$$(6.3) \qquad = \frac{udS - rS_h(\theta)}{r(u-d)dS} P_{dd} < 0$$

because $S_h(\theta_1) \geq S_h(\theta = 0) > \frac{ud}{r} S$. We now need to see that. $S_h(\theta_1) - S_\ell(\theta_1) > 0$. We first see the $A_\ell - A_1(\theta = 0) < 0$ Indeed,

$$A_\ell - A_1 = A_\ell - \left(\frac{-P_\ell}{S_h - S_\ell} \right) = A_\ell + \frac{A_\ell S_\ell + B_\ell}{S_h - S_\ell} = \frac{A_\ell S_h + B_\ell}{S_h - S_\ell} \tag{6.4}$$

But we have already seen above that $A_\ell S_h(\theta = 0) + B_\ell < 0$ and $S_h(\theta = 0) - (S_\ell)(\theta = 0) > 0$, so that $A_\ell - A_1(\theta = 0) < 0$. We then have that $= A_\ell - A_1(\theta = 0) < 0 \frac{\partial}{\partial \theta}(\theta A_\ell)\Big|_{\theta=0}$ and hence $\dot{S}_\ell(\theta = 0) < 0$.

Suppose that there exists $\theta_2 > 0$ such that $\dot{S}_\ell(\theta_2) = 0$. Then $\frac{\partial}{\partial \theta}(\theta A_1)\Big|_{\theta_2} = A - A_1(\theta_2) - \theta_2 \frac{\partial A_1}{\partial \theta}\Big|_{\theta_2}$

$$(A_\ell - A_1)(S_h - S_\ell)^2 = -\theta_2 A_\ell \dot{S}_\ell (S_h - S_\ell) + \theta_2 (A_\ell S_\ell + B_\ell)(\dot{S}_h - S_\ell)$$

$$(A_\ell - A_1)(S_h - S_\ell)^2 = -\theta_2 (A_\ell S_\ell + B_\ell) \dot{S}_h = -\theta P_\ell \dot{S}_h \tag{6.5}$$

because $\dot{S}_\ell(\theta_2) - 0$ But (6.4) implies that $\dot{S}_h(\theta_2) < 0$, which is a contradiction.

8. Case III.

Suppose now that all agents know that the amount θ of insurance is a random variable with distribution μ but they don't know the exact value of θ.

The amount $D_j, j = h, \ell$, of stock that market makers want to hold from period 2 to period 3 remains the same:

$$D_j(S_j) = \frac{1}{a(u-d)S_j^*}\log\left(\frac{p(S_{ju} - rS_j)}{(1-p)(rS_j - S_{jd})}\right) \qquad S_j = \begin{cases} uS & j = h \\ dS & l = \ell \end{cases}$$

For period 1 they maximise the expected profits $f(Z_1, D_h, D_\ell) = \max_x f(x, D_h, D_\ell)$, where

$$f(x, D_h, D_\ell) = \int \sum_{j,k} p_{jk} \mathcal{U}\left(\omega + xr(S_j(\theta) - rS)\right) + D_j(S_j(\theta))\left(S_{jk} - rS_j(\theta)\right) d\mu(\theta)$$

where $U(y) = -e^{-ay}$ and $S_j(\theta), j = h, \ell$ are given by the equilibrium equations at period two 2:

$$Z_1 = \arg \max_x f\ (x, D_h, D_\ell\ )$$

$$Z_h(S_h(\theta)) = D_h(S_h(\theta)) - Z_1 = -\theta(A_h - A_1)$$

$$Z_\ell(S_\ell(\theta)) = D_\ell(S_\ell(\theta)) - Z_1 = -\theta(A_\ell - A_1)$$

$$A_h = 0 \qquad , \qquad A_\ell = -\frac{K - d^2 S}{(u-d)dS}$$

and A_1 is supposed to be[9].

9 Let this hypothesis pass for a moment. It corresponds to the strategy which gives the expected P1, i.e.

$$\int P_1(\theta)d\mu(\theta) = S\int A_1(\theta)d\mu(\theta) + \int B_1(\theta)d\mu(\theta).$$

The problem is that the market is no longer complete and hence A1 can not be computed accurately any more. Our point now is that no matter how A1 is calculated, (P1 can not be accurately computed) and the price S^ℓ (θ) decreases with θ.

$$A_1 = \int A_1(\theta)d\mu(\theta)$$

$$A_1(\theta) = \frac{P_h - P_\ell}{S_h(\theta) - S_\ell(\theta)} = \frac{-A_\ell S_\ell(\theta) - B_\ell}{S_h(\theta) - S_\ell(\theta)}$$

$$A_\ell = \frac{K - d^2 S}{(u-d)dS} \quad , \quad B_\ell = \frac{K - d^2 S}{(u-d)r}$$

Suppose that this system of equations has a solution for Z_1. The equilibrium equations for period 2 are

$$D_j(S_j) = Z_j - \theta A_j$$

$$\frac{p(S_{ju} - rS_j)}{(1-p)(rS_j - S_{jd})} = \exp\left(Z_1 a(u-d)S_j^*\right) \exp\left(-\theta A_j a(u-d)S_j^*\right)$$

$$S_k(\theta) = \frac{1}{r}\left(\frac{pu + (1-p)\Gamma_j(\theta)d}{p + (1-p)\Gamma_j(\theta)}\right)S_j^*, \qquad k = u, d$$

$$\Gamma_k(\theta) = R^j \exp\left(-\theta a(u-d)S_j^* A_k\right)$$

$$A_k := A_k - A_1 \quad , \quad R := \exp\left(a(u-d)Z_1 S\right) > 0$$

Observe that by (4.4) and (6.4) we always have that $A_\ell < A_1(\theta) < 0$ for all θ, therefore $A_\ell < A_1 < 0$. Comparing these equations with those of (4.1), since $R > 0$, we get that:

$$\frac{\partial S_h}{\partial \theta} > 0 \quad \text{and} \quad \frac{\partial S_\ell}{\partial \theta} < 0 \quad \text{for all} \quad \theta \geq 0;$$

Moreover, suppose there exists a number[10] $\hat{\theta}>0$ (for exemplo $\hat{\theta}=0$) such that the rates u, d, r satisfy

$$S_h(\hat{\theta})=uS \quad , \quad S_\ell(\hat{\theta})=dS$$

Then

$$\left.\frac{\partial P_1}{\partial\theta}\right|_\theta>0 \quad \text{for all } \hat{\theta}\geq 0.$$

9. Case IV

In this case the portfolio insurers are supposed to know θ. The market makers do not know θ but they assume that θ has a probability distribution μ on the interval [0, 1].

The gross demand of the market makers, Z_1, D_h (S_h), D_ℓ (S_ℓ), are obtained as in case III. Their net demands are Z_1, Z_h $(S_h) = D_h(S_h) - Z_1$, $Z_h = D_\ell$ $(S_\ell) - Z_1$. In this case we assume that the random variables α and θ are related by the formula (6.0) where Δ is a constant, so that the price S of the stock at period I does not reveal the value of θ.

The demands of portfolio insurers are obtained as in case II:

$$A_1(\theta)=\frac{-A_\ell S_\ell(\theta)-B_\ell}{S_h(\theta)-S_\ell(\theta)}$$

$$A_h=0A_\ell \quad , \quad =-\frac{K-d^2S}{(u-d)dS}$$

$$Y_j=\theta\left(A_j-A_1\right) \quad , \quad j=h,\ell$$

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10 For example, by (4.4), this condition holds for $\hat{\theta}$ = 0 if ud > r2.

$$\frac{\partial S_h}{\partial \theta} > 0 \quad , \quad \frac{\partial S_\ell}{\partial \theta} < 0 \qquad \text{for all} \quad \hat{\theta} \geq 0$$

Note that in this case the market is complete from the viewpoint of the portfolio insurers. They know the value of their strategies, but the volatility of the stock increases due to their presence in the market.

10. Case V

In this case the value of θ is public known. This case corresponds to the existence of a real put, if we assume that all the suppliers of the real put synthesise it in order to cover their position. Then the value of θ is seen as the amount of written puts on the market.

In this case the solution to the maximisation problem (3.2) for the market is

$$Z_1 = \frac{1}{ar(S_h - S_\ell)} \log\left(\frac{p(S_h - rS)B_h}{(1-p)(rS - S_\ell)B_\ell}\right)$$

where

$$B_j := \sum_{k=u,d} p_k e^{-aD_j(S_k - rS_j)}, \qquad j = h, \ell \,;$$

$D_j = D_j(S_j)$ are given by (3.1) and $S_h = S_h(\theta), S_\ell = S_\ell(\theta)$ are given by the equilibrium equations with $A_1(\theta)$ as in cases II or IV.

The prices at period II are given by

$$S_j(\theta) = \frac{1}{r}\left(\frac{pu + (1-p)\Lambda_j(\theta)d}{p + (1-p)\Lambda_j(\theta)}\right)S^* j, \quad j = h, \ell$$

$$\Lambda_j(\theta) = \exp\left(a(u-d)Z_1(\theta)S\right)\exp\left(-\theta a(u-d)S^*{}_j A_j\right)$$

$$A_j = A_j - A_1(\theta) \quad , \quad S_h^* = uS \quad , \quad S^*{}_\ell = dS.$$

This case becomes difficult to analyse. Nevertheless we can do some comparisons between the different cases. The equations are

Cases I, III.

$$D_h^I(S_h) = Z_1 + \theta A_1$$
$$D_\ell^I(S_\ell) = Z_1 + \theta A_1 - \theta A_\ell$$

Cases II, IV.

$$D_h^{II} = Z_1 + \theta A_1(\theta)$$
$$D_\ell^{II} = Z_1 + \theta A_1 - \theta A_\ell$$

Case V.

$$D_h^V = Z_1(\theta) + \theta A_1(\theta)$$
$$D_\ell^V = Z_1(\theta) + \theta A_1(\theta) - \theta A_\ell$$

On cases I-IV, we have that

$$\lim_{\theta \to +\infty} S_h(\theta) = \frac{u^2}{r} S,$$

$$\lim_{\theta \to +\infty} S_\ell(\theta) = \frac{d^2}{r} S.$$

These are the upper and lower bounds for all the possible prices of the stock if it is supposed that there can not be arbitrage profits above the rate of interest of the economy.

From the equilibrium equations above we see that for cases (I, II) or (III, IV) we have that either

$$S_h^I > S_h^{II}(\theta) \quad \text{and} \quad S_\ell^I(\theta) > S_\ell^{II}(\theta) \qquad \text{for all } \theta \geq \theta o.$$

or

$$S_h^I(\theta) < S_h^{II}(\theta) \quad \text{and} \quad S_\ell^I(\theta) < S_\ell^{II}(\theta) \text{ for all } \theta > \theta o.$$

This depends on the value of $\frac{\partial A_1}{\partial \theta}$, but this is unclear. For example, we have that

$$A_1 = -\frac{A_\ell S_\ell + B_\ell}{S_h - S_\ell}$$

and then

$$-\frac{\partial A_1}{\partial \theta}(S_h - S_\ell)^2 = (A_\ell S_h + B_\ell)\dot{S}_\ell - (A_\ell S_\ell + B_\ell)\dot{S}_h.$$

In this equation $A_\ell S_h + B_\ell < 0, \dot{S}_\ell < 0, A_\ell S_\ell + B_\ell = P_\ell > 0$ and $S_h > 0$. This does not give a definite sign for $\frac{\partial A_1}{\partial \theta}$.

If we try to compare cases IV (or II) and V, we need to know the behaviour of $Z_1(\theta)$ with θ. This is not clear to us. The portfolio insurers react ($A_1(\theta)$) to the " volatility" of the stock compared to how deep the price $S_\ell(\theta)$ can decrease. The market makers react to the exponential of their *expected* profits. If for a large θ (say $\theta > \theta_o$) the market makers find it more profitable to buy more stock on period 1 (i.e. $Z_1(\theta) > Z_1(\theta o)$) then we will have that

$$S_h^V < S_h^{II}(\theta) \quad \text{and} \quad S_\ell^V < S_\ell^{II}(\theta).$$

If the market makers find it more profitable to reserve money form period one in order to make transactions on period 2 (i.e. $Z_1(\theta) < Z_1(\theta_o)$), then we will have that

$$S_h^V > S_h^{II}(\theta) \quad \text{and} \quad S_\ell^V > S_\ell^{II}(\theta).$$

II. Existence of a Real Put

In this case we have two separated markets, the stocks market and the options market.

We can analyse three different situations:

[a] If the suppliers of the put at the options market synthesise all the puts that they sell. This corresponds to case V as seen above.

[b] If the suppliers of the put assume the risk of all the puts that they sell. This corresponds to case I with $\theta = \theta_o = 0$, because the buyers of *real* put options behave in the stocks market as buy and hold agents. Nothing unexpected happens in the stocks market.

[c] If the suppliers of the put assume the risk of a fraction β of the puts sold and synthesise a fraction (1 - β) of the puts sold at the options market. Suppose that ' β is not known by the agents. Let® be the amount of puts being synthesised. This situation corresponds to case I with a new equation for the first period:

$$(\alpha - 1) + \theta A_1 + Z_1(S) = 0 \ ,$$

where the buyers of real puts are seen as buy and hold agents and $\theta = (1 - \beta) \times E$, E = number of real puts sold at the options market. This change doesn't modify neither the other equations nor the conclusions of case I. The expected value θ_o of θ can be found by the equation $P_1\ \theta$ = Price of the (real) put in the options market at period 1, where $P_1\ \theta$ is given by formula (5.1).

Another possibility is that agents use a probability distribution for β. This corresponds to case IV with $\theta = (1 - \beta) \times E$. This probability should satisfy: Expected value $[P_1(\theta)]$ = Price of the real put. If β is known, the situation corresponds to case V.

12. REFERENCES

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